Delocalization power of global unitary operations on quantum information

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Abstract. We investigate how two pieces of localized quantum information represented by a tensor product of two unknown qudit states are delocalized by performing two-qudit global unitary operations. To characterize the delocalization power of global unitary operations on quantum information, we analyze the necessary and sufficient condition to deterministically relocalize one of the two pieces of quantum information to its original Hilbert space by using only local operation and classical communication (LOCC). We prove that this LOCC one-piece relocalization is possible if and only if the global unitary operation is local unitary equivalent to a controlled-unitary operation. The delocalization power and the entangling power characterize different non-local properties of global unitary operations.

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1. Introduction

Quantum information processing (QIP) aims to achieve more efficient information processing or to perform tasks that are not classically possible, by utilizing the superposition principle of quantum states. Extending our knowledge of what QIP can provide is a driving force of developments in quantum technologies and this also reveals the fundamental difference between quantum and classical systems. Understanding what makes QIP outperform its classical counterpart is essential for understanding yet unknown potentials of QIP.

Information processing can be interpreted as operations mapping from input information into output information in both classical and quantum cases. In both cases, operations over several bits or qubits are essential for information processing. In QIP, operations over several qubits are called global (or non-local) operations. Global operations can generate entanglement, one of the most distinguishable characters of quantum mechanics from classical mechanics, whereas local operations over single qubits cannot generate entanglement even adding classical correlations. Entanglement has been investigated in the local operation and classical communication (LOCC) paradigm [1].

Although the entanglement of quantum states provides many clues for understanding the superiority of QIP [2]–[4], it has also turned out that entanglement alone is not sufficient for understanding every aspect of QIP. The Gottesman–Knill theorem [5] implies that classical computers can efficiently simulate a certain type of QIP generating entanglement. On the other hand, in a model of quantum computation called the deterministic quantum computation with one pure quantum bit (DQC1) [6], there are some cases where quantum speedup can be achieved without generating entanglement. We need another clue for understanding the advantages of QIP.

One of the essential differences in QIP and the classical counterpart is that QIP sometimes involves global operations on unknown input states, namely, arbitrary superpositions of quantum states where their superposition coefficients are unknown. Quantum teleportation [7] and quantum error corrections [8] are typical examples. We call such unknown states quantum information in this paper. Quantum information cannot be measured (i.e. estimating unknown coefficients by finite measurements) perfectly and cannot be copied perfectly either [9]. In contrast, the classical information in QIP can be encoded in a set of known orthogonal states and it can be perfectly distinguished. Classical information can also be obtained by the result of measurements in QIP. QIP can be analyzed by investigating how input quantum information is transformed to output quantum information due to global operations. Therefore, evaluation of the effects of the global operation on quantum information is desired.

For evaluation, the simplest global operations in QIP are global unitary operations. One way of evaluating global unitary operations is by their entangling power [10]. The entangling power is the maximum amount of entanglement generated by a unitary operation among all possible quantum input states. It gives a way of quantifying global unitary operations and is useful for finding the lower bound of the entanglement resource required to perform the unitary operation between distributed parties. However, it evaluates the maximum value achieved for certain states, and its operational meaning in terms of quantum information is not clear.

In this paper, we present an approach for this problem by introducing the concept of delocalization of initially localized two ‘pieces’ of quantum information by a global unitary operation $U$ and investigating the power of delocalization. To characterize how ‘powerful’ the delocalization effect is on two pieces of quantum information caused by the global unitary operation, we introduce another concept called relocalization. Relocalization is a process that
deterministically transforms at least one of the pieces of delocalized quantum information into its minimal Hilbert space. Our idea is to evaluate the power of delocalization of global unitary operations by analyzing non-local properties of the quantum operations required for relocalizing one of the two pieces of delocalized quantum information.

We define two classes of global unitary operations on the two-qudit (quantum $d$-level) Hilbert space in terms of the delocalization power characterized by one-piece relocalization. For the first class of global unitary operations, one piece of quantum information can be relocalized by only using LOCC. For the second class, one-piece relocalization cannot be performed by LOCC, namely, it requires global operations. In this paper, we show that the first class consists of controlled-unitary operations and their local unitary equivalent operations. Other global unitary operations fall into the second class. Since the second class requires more non-local resources than the first class, our classification also gives a way of ranking the delocalization power of global unitary operations.

We note that a related problem has been investigated by Gregoratti and Werner [11] (quantum lost and found). Their problem can also be interpreted to find a condition for delocalized quantum information to be relocalized by just using LOCC. In their setting, one of the two qudits is chosen as the environment and the initial state of the environment is fixed as a known pure state. It corresponds to a special situation of our case where only one piece of quantum information is delocalized in the two-qudit Hilbert space. In their problem, measurements in LOCC are only allowed on the environmental qudit. This restriction on the measurement is equivalent to restricting general LOCC to more limited one-way LOCC. They investigated the condition on the completely positive and trace-preserving (CPTP) map of the non-environmental qudit alone, not global unitary operations, for the delocalized piece of quantum information to be relocalizable by LOCC. They showed that the necessary and sufficient condition of the CPTP map is that the map is a random unitary channel. Ogata and Murao [12] have also considered a related problem to find a necessary and sufficient condition for a restricted class of delocalized two pieces of qubit quantum information in the two-qubit Hilbert space to be relocalized by LOCC. In their work, they considered general LOCC, but their LOCC operations required a partial knowledge of one of the qubit states. Thus, the main difference between these previous works and our work is that we treat full two pieces of input quantum information to evaluate the delocalization power of global unitary operations.

This paper is organized as follows. In section 2, we define the delocalization and relocalization of quantum information and introduce the classification of the two-qudit global unitary operations in terms of the delocalization power evaluated by LOCC one-piece relocalizability. In section 3, we show that if a general LOCC one-piece relocalization protocol exists, then it can be reduced to one-way LOCC. In section 4, we show that if a one-way LOCC one-piece relocalization protocol using a general measurement exists, then there also exists a one-way LOCC protocol using a projective measurement. We derive the necessary and sufficient condition for two-qudit global unitary operations to be LOCC one-piece relocalizable. In section 5, we give the conclusion and discussions.

2. Delocalization and relocalization

Before introducing delocalization and relocalization, we present our notion of ‘pieces’ of quantum information. We consider an unknown state of a qudit described by $|\psi\rangle = \sum_{i=0}^{d-1} \alpha_i |i\rangle$, where $\{|i\rangle\}$ is a basis of the $d$-dimensional Hilbert space $\mathcal{H} = \mathbb{C}^d$ and $\{\alpha_i\}$ is a set of arbitrary
coefficients normalized by $\sum |\alpha_i|^2 = 1$. We interpret this unknown state as a situation where a piece of quantum information of a qudit is localized in its minimal Hilbert space $\mathcal{H}$.

We consider a situation where each one of two parties named Alice and Bob possesses a piece of quantum information. Let $\mathcal{H}_A$ and $\mathcal{H}_B$ denote the Hilbert space describing Alice’s and Bob’s qudit, respectively. The joint state describing the two pieces of quantum information is given by a tensor product of two qudit states $|\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$, where $|\psi_A\rangle \in \mathcal{H}_A$ and $|\psi_B\rangle \in \mathcal{H}_B$. Note that, by definition of calling the state to be quantum information, Alice and Bob do not know the state, although we assume they know that the state $|\psi_{AB}\rangle$ consists of a tensor product of two pure states.

Now we introduce the delocalization of quantum information. In this paper, we use the word global unitary operations to denote unitary operations acting globally on the two-qudit Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, not including the tensor products of single-qudit unitary operations. Performing a two-qudit global unitary operation $U$ on a tensor product state $|\psi_{AB}\rangle$ generally gives an entangled state $U|\psi_A\rangle \otimes |\psi_B\rangle$ in the joint Hilbert space $\mathcal{H}_{AB}$. The reduced density matrices $\rho_A = \text{Tr}_B|\psi_{AB}\rangle \langle \psi_{AB}|$ and $\rho_B = \text{Tr}_A|\psi_{AB}\rangle \langle \psi_{AB}|$ are no longer equal to their respective initial states $|\psi_A\rangle$ and $|\psi_B\rangle$. We consider that both pieces of quantum information are delocalized over the same joint Hilbert space due to the global unitary operation $U$. We note that after the swap operation $U_{\text{SWAP}}$, whose action is given by $U_{\text{SWAP}} |\psi_A\rangle \otimes |\psi_B\rangle = |\psi_B\rangle \otimes |\psi_A\rangle$ for all $|\psi_A\rangle$ and $|\psi_B\rangle$, the two pieces of quantum information are localized in the minimal Hilbert space, but they are no longer contained in their original Hilbert space. Thus, we consider that quantum information is delocalized by the swap operation in our definition. In a sense, delocalization is a type of non-local effect induced by global unitary operations.

Relocalization is a process that deterministically transforms at least one of the pieces of delocalized quantum information over $\mathcal{H}_{AB}$ into its original Hilbert space $\mathcal{H}_A$ for Alice’s piece of quantum information $|\psi_A\rangle$ or into $\mathcal{H}_B$ for Bob’s piece of quantum information $|\psi_B\rangle$. To relocalize two pieces of delocalized quantum information perfectly to their original form, we need to perform the conjugate unitary operation $U^\dagger$. However, if we only require one piece of delocalized quantum information to be relocalized perfectly, this one-piece relocalization can be done by more general quantum operations described by CPTP maps, not necessarily the global unitary operation $U^\dagger$.

In the rest of the paper, we analyze the case when Bob’s piece of quantum information $|\psi_B\rangle$ is to be relocalized to $\mathcal{H}_B$. Denoting the CPTP map of a deterministic one-piece relocalization for a global unitary operation $U$ to be $\Gamma_U$, its action is given by

$$\Gamma_U \left( U |\psi_A\rangle \langle \psi_B| \otimes |\psi_B\rangle \langle \psi_B| U^\dagger \right) = \rho_A \otimes |\psi_B\rangle \langle \psi_B|$$

for arbitrary $|\psi_A\rangle$ and $|\psi_B\rangle$, where $\rho_A \in \mathcal{S}(\mathcal{H}_A)$ is a density matrix independent of $|\psi_B\rangle$.

We analyze how strong the delocalization power of the global unitary operation $U$ is on quantum information, by investigating how non-local the one-piece relocalization CPTP map $\Gamma_U$ should be. Particularly, in this paper, we classify the global unitary operations $U$ by investigating the necessary and sufficient condition for a one-piece relocalization CPTP map $\Gamma_U$ to be achieved by LOCC. All global unitary operations can be divided into the following two classes:

Class 1. Global unitary operations that are one-piece relocalizable by LOCC.

Class 2. Global unitary operations that are not one-piece relocalizable by LOCC.

These classes give a ranking of global unitary operations by their delocalization power.
Figure 1. Schematic picture of the concept of delocalization and LOCC one-piece relocalization. The rectangle on the top consists of two squares, representing the two-qudit Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. The left square with vertical lines represents Alice’s Hilbert space $\mathcal{H}_A$, in which a pure state $|\psi_A\rangle$ is localized. Similarly, the right square with the vertical lines represents Bob’s Hilbert space $\mathcal{H}_B$, with $|\psi_B\rangle$ being localized. The arrows on the left-hand side indicate the delocalization of the two pieces of quantum information by a class 1 global unitary operation $U_1$ followed by the one-piece relocalization of Bob’s quantum information using a LOCC map $\Gamma_{\text{LOCC}}$. The arrows on the right-hand side correspond to the delocalization by a class 2 global unitary operation $U_2$, which is relocalized by a global CPTP map $\Gamma_{\text{global CPTP}}$. An appropriate entangled resource must be supplied on top of LOCC to implement $\Gamma_{\text{global CPTP}}$. In both cases, the two pieces of localized quantum information spread across the whole two-qudit Hilbert space (becoming ‘intertwined’ with each other) and Alice’s piece of quantum information is sacrificed.

Showing that a given global unitary operation belongs to class 1 is relatively easy. We just need to show a construction of a LOCC protocol for one-piece relocalization. On the other hand, it is more difficult to prove that a given global unitary operation belongs to class 2, because it must be shown that one-piece relocalization is impossible by any LOCC protocol (figure 1).

We prove this classification problem in the following manner. First, we derive a necessary condition on the LOCC protocol for one-piece relocalization for a particular input state by introducing accumulated operators representing LOCC. We show that this condition restricts the operators representing the protocol and this restriction implies that any LOCC one-piece relocalization protocol can be reduced to a one-way LOCC protocol. Such one-way LOCC
protocols are shown to be decomposed into Alice’s general measurement and Bob’s local unitary operation conditional to Alice’s measurement outcome. By reducing the one-way LOCC protocol with a general measurement into one with a projective measurement, we prove that a global unitary operation is one-piece relocalizable if and only if it is a controlled-unitary operation and its local unitary equivalent, where the control qudit is Alice’s and the target is Bob’s.

3. Reduction to one-way local operation and classical communication (LOCC)

We denote the LOCC one-piece relocalization CPTP map for a global unitary operation $U$ as $\Gamma^L_U$. Suppose that each of Alice and Bob has an extra ancilla qudit. Although the relocalization CPTP map is only required to relocalize one piece of quantum information, the linearity of CPTP maps guarantees that such a relocalization map also recovers the state of one of the two input qudits even when each of the qudits is entangled with the ancilla qudit.

Assume that Alice’s input qudit is maximally entangled with an ancilla qudit and Bob’s input qudit is also maximally entangled with another ancilla qudit. The reduced density matrix of the joint state of Alice and Bob’s input qudits is given by the tensor product of two completely mixed states, which is described by $\mathbb{I}/d \otimes \mathbb{I}/d$, where $\mathbb{I}$ denotes the identity operator on $d$-dimensional Hilbert space. Then the action of $\Gamma^L_U$ should be given by

$$\Gamma^L_U \left( U \left( \frac{\mathbb{I}}{d} \otimes \frac{\mathbb{I}}{d} \right) U^\dagger \right) = \rho_A \otimes \frac{\mathbb{I}}{d}$$

(2)

for some density matrix $\rho_A$. The left-hand side of (2) can be simplified because the tensor product of the identity operators commutes with the global unitary operator $U$. Since the CPTP map is independent of $U$ for this special initial state, we denote $\Gamma^L_U \equiv \Gamma^L_{U=\mathbb{I}}$ and simplify the equation to

$$\Gamma^L \left( \frac{\mathbb{I}}{d} \otimes \frac{\mathbb{I}}{d} \right) = \rho_A \otimes \frac{\mathbb{I}}{d}.$$  

(3)

In general, LOCC protocols consist of a series of local measurements on the local Hilbert space conditional on the classical information of the measurement outcomes of the previous local measurements, except for the first turn and the last turn of LOCC, where the first local measurement is not conditional on the previous measurements by definition, and it is not necessary for the last turn to be finished by a measurement, but a unitary operation.

We describe a LOCC protocol by using a set of ‘accumulated’ operators consisting of a tensor product of Alice’s accumulated operations $M^{\vec{n}_i}$ acting on $\mathcal{H}_A$ and Bob’s accumulated operations $K^{\vec{n}_i}$ acting on $\mathcal{H}_B$, where the superscript $\vec{n}_i$ indicates the dependency on all the measurement outcomes up to the $i$th turn. The measurement outcome of the $k$th $(1 \leq k \leq i)$ turn is represented by the $k$th component $n_k$ of the vector $\vec{n}_i = (n_1, n_2, \ldots, n_i)$. The set of vectors $\{\vec{n}_i\}$ denotes all possible sequences of measurement outcomes that Alice and Bob have obtained up to the $i$th turn. Note that $\vec{n}_i$ indicates a ‘branch’ of measurement outcomes, and does not only indicate the measurement outcome of the $i$th turn. The vector $\vec{n}_i$ and its element $n_i$ should not be confused.

By denoting the conditional measurement operator for the $i$th measurement performed by Bob for a measurement branch $\vec{n}_i$ to be $K^{\vec{n}_i|\vec{n}_{i-1}}$, the relationship between two accumulated operators $K^{\vec{n}_i}$ and $K^{\vec{n}_{i-1}}$ is given by

$$K^{\vec{n}_i} = K^{\vec{n}_i|\vec{n}_{i-1}} K^{\vec{n}_{i-1}}.$$  

(4)
A similar relation holds for $\vec{M}_i$ if the $i$th measurement is performed by Alice. By iterating this procedure, the accumulated operators can be decomposed to products of conditional local measurements alternatingly performed by Alice and Bob, and this gives a standard notation of LOCC [13].

Using the accumulated operators, a general $N$-turn LOCC maps a density matrix $\rho_{AB} \in S(\mathcal{H}_A \otimes \mathcal{H}_B)$ to

$$\rho_{AB} \rightarrow \sum_{\vec{n}_N} (M_{\vec{n}_N} \otimes K_{\vec{n}_N}) \rho_{AB} (M_{\vec{n}_N} \otimes K_{\vec{n}_N})^\dagger.$$  \hspace{1cm} (5)

To guarantee the trace-preserving property, the completeness relation of the accumulated operators $M_{\vec{n}_i}$ and $K_{\vec{n}_i}$, namely,

$$\sum_{n_i} M_{\vec{n}_i}^\dagger M_{\vec{n}_i} = M_{\vec{n}_{i-1}}^\dagger M_{\vec{n}_{i-1}}$$  \hspace{1cm} (6)

and

$$\sum_{n_i} K_{\vec{n}_i}^\dagger K_{\vec{n}_i} = K_{\vec{n}_{i-1}}^\dagger K_{\vec{n}_{i-1}},$$  \hspace{1cm} (7)

should be satisfied for all $1 \leq i \leq N$.

Returning to LOCC one-piece relocalization, its $N$-turn LOCC CPTP map $\Gamma^L$ given by (3) can be represented by using the accumulated operators by

$$\Gamma^L \left( \frac{1}{d} \otimes \frac{1}{d} \right) = \sum_{\vec{n}_N} (M_{\vec{n}_N} \otimes K_{\vec{n}_N}) \left( \frac{1}{d} \otimes \frac{1}{d} \right) (M_{\vec{n}_N} \otimes K_{\vec{n}_N})^\dagger$$

$$= \frac{1}{d^2} \sum_{\vec{n}_N} M_{\vec{n}_N} M_{\vec{n}_N}^\dagger \otimes K_{\vec{n}_N} K_{\vec{n}_N}^\dagger$$

$$= \rho_A \otimes \frac{1}{d} I.$$  \hspace{1cm} (8)

Since Bob’s qudit should be maximally entangled to his ancilla after the one-piece relocalization,

$$K_{\vec{n}_N} K_{\vec{n}_N}^\dagger \propto I$$  \hspace{1cm} (9)

should be satisfied for all possible branches of outcomes $\vec{n}_N$. Therefore, Bob’s accumulated operator has to be proportional to a unitary operation and can be written as $K_{\vec{n}_N} = \sqrt{p_{\vec{n}_N}} u_{B_{\vec{n}_N}}$ for all $\vec{n}_N$, where $p_{\vec{n}_N}$ is a real number satisfying $0 \leq p_{\vec{n}_N} \leq 1$, $\sum_{\vec{n}_N} p_{\vec{n}_N} = 1$ and $u_{B_{\vec{n}_N}}$ is a local unitary operator. This Bob’s accumulated operator can be interpreted to perform a random unitary operation $u_{B_{\vec{n}_N}}$ depending on the branch $\vec{n}_N$ with probability $p_{\vec{n}_N}$.

By using (7), we can show that $K_{\vec{n}_{i-1}}^\dagger K_{\vec{n}_{i-1}} \propto I$, and by iterating this procedure we obtain $K_{\vec{n}_i} \propto u_{B_{\vec{n}_i}}$ for all $1 \leq i \leq N$. Then Bob’s conditional measurement operator $K_{\vec{n}_i|\vec{n}_{i-1}}$ for all $\vec{n}_i$ and $i$ should also be proportional to a unitary operator. Thus, all Bob’s operations in LOCC one-piece relocalization have to be random unitary operations that depend on the sequence of the measurement outcomes up to the previous turn.

When Bob’s operation in a LOCC protocol at a particular turn is a random unitary operation, Alice’s measurements in the subsequent turns do not depend on this outcome of Bob. In this case, Bob does not need to send any classical information to Alice. Therefore, any LOCC one-piece relocalization protocol does not require the communication from Bob to Alice, and can be reduced to a one-way LOCC protocol.
4. Relocalization by one-way LOCC

By redefining the combination of multiple measurements on Alice’s qudit as a single measurement, we can reduce our problem to a one-way LOCC protocol using only one measurement. Then the accumulated operator of the one-way LOCC one-piece relocalization protocol for a global unitary operator $U$ is given by

$$M_U^n \otimes u_U^n,$$

where $M_U^n$ is a measurement operator corresponding to the measurement outcome $n$ and $u_U^n$ is Bob’s conditional local unitary operation. Note that a vector superscript $\tilde{n}$ reduced to a single variable $n$ denoting Alice’s measurement outcome in the first turn.

Using this accumulated operator, the action of the one-way LOCC CPTP map of (deterministic) one-piece relocalization has to satisfy

$$(M_U^n \otimes u_U^n)U |\psi_A\rangle \otimes |\psi_B\rangle |\psi_B\rangle \langle \psi_B| U^\dagger (M_U^n \otimes u_U^n)\dagger$$

$$= \rho_{A,U}^{n} \otimes |\psi_B\rangle \langle \psi_B|$$

for all $n$. Since $|\psi_A\rangle$ and $|\psi_B\rangle$ can be arbitrary, $(M_U^n \otimes u_U^n)U = M_U^n \otimes I$ should hold, where $M_U^n$ is an operator satisfying $M_U^n |\psi_A\rangle \langle \psi_A| M_U^n \dagger = \rho_{A,U}^{n}$. Then we have $M_U^n M_U^n = M_U^n M_U^n \dagger$, which implies that $M_U^n = M_U^n \otimes v_U^n$ with an appropriate local unitary operator $v_U^n$ of Alice. Therefore, we obtain a condition for the LOCC one-piece relocalization given by

$$(M_U^n \otimes u_U^n)U = (M_U^n \otimes v_U^n) \otimes I.$$

In general, it is not necessary for the measurement operators $M_U^n$ in a one-way LOCC to be Hermitian. However, for our analysis of LOCC one-piece relocalization, we can always take Hermitian measurement operators without loss of generality. To see this, we denote the polar decomposition of $M_U^n$ by

$$M_U^n = w_U^n |M_U^n|,$$

where $|M_U^n| = \sqrt{M_U^n \dagger M_U^n}$ and $w_U^n$ is a unitary operator. Then any non-Hermitian measurement operator $M_U^n$ can be represented by a Hermitian measurement operator $|M_U^n|$ followed by an appropriate local unitary operation $w_U^n$. The additional local unitary operation $w_U^n$ does not affect the classification in terms of LOCC one-piece relocalization.

The Hermitian measurement operators $M_U^n$ can be written as

$$M_U^n = \sum_k m_U^{n,k} P_U^{n,k},$$

where $m_U^{n,k}$ represents an eigenvalue of $M_U^n$ and $P_U^{n,k}$ is a rank-1 projector whose range is the eigenspace corresponding to $m_U^{n,k}$. By multiplying $P_U^{n,k} \otimes u_U^n$ from the left on (12) and using (13) and (14), we obtain

$$(P_U^{n,k} \otimes I)U = (P_U^{n,k} v_U^n \otimes u_U^n).$$

Taking the summation over $k$ and introducing a projector given by $P_U^n = \sum_k P_U^{n,k}$ whose range is the support of $M_U^n$, we obtain a simplified condition of LOCC one-piece relocalization given by

$$(P_U^n \otimes I)U = (P_U^n v_U^n \otimes u_U^n).$$
By multiplying its complex conjugate of (16) from the right, we have

\[ P_U^n P_U^m \otimes I = (P_U^n v_U^n v_U^m^\dagger P_U^m) \otimes (u_U^n u_U^m^\dagger). \]  

(17)

First, we consider a special case where Alice’s measurement is given by a projective measurement satisfying \( M_U^n = P_U^n \) and \( P_U^n P_U^m = \delta_{n,m} P_U^n \). In this case, from (17), the condition for the operators acting on Alice’s Hilbert space is given by \( P_U^n v_U^n v_U^m^\dagger P_U^m = \delta_{n,m} I \). This condition implies that there is a unitary operator \( v_U \) satisfying \( P_U^n v_U^n = P_U^n v_U^n \) and \( P_U^m v_U^m = P_U^m v_U^m \). On the other hand, the unitary operators acting on Bob’s Hilbert space \( u_U^n u_U^m \) do not have any restriction. Thus we rewrite the condition (16) as

\[ (P_U^n \otimes I) U = (P_U^n v_U^n) \otimes u_U^n, \]  

(18)

where Alice’s unitary operator \( v_U \) does not depend on \( n \) anymore.

By taking summation over \( n \) of (18), and using the completeness relation of the projectors \( \sum_n P_n = I \), we obtain the condition for a global unitary operation

\[ U = \left( \sum_n P_n^n \otimes u_U^n \right) v_U \otimes I. \]  

(19)

The first part of the right-hand side, namely,

\[ \sum_n P_n^n \otimes u_U^n, \]  

(20)

denotes a controlled-unitary operation where the local unitary operation \( u_U^n \) is performed on Bob’s qudit when the state of Alice’s qudit is determined by the projector \( P_U^n \). Therefore, the global unitary operation \( U \) should be local unitary equivalent to the controlled-unitary operation given by (20).

Next, we consider a more general case where Alice’s measurement is a non-projective measurement. In the following, we show that if \( U \) is LOCC one-piece relocatable by a non-projective measurement, then there always exists a relocalization protocol using a projective measurement. Thus, we can reduce a one-way LOCC protocol with a general measurement to one with a projective measurement.

We consider two projectors \( P_U^n \) and \( P_U^m \), where \( P_U^n P_U^m \neq 0 \) for \( n \neq m \). Then the operator \( u_U^n u_U^m \) acting on Bob’s Hilbert space in (17) has to be equivalent to \( I \) up to a global phase. We set \( u_U^m = e^{i\theta_{n,m}} u_U^n \) by choosing an angle \( \theta_{n,m} \) where \( 0 \leq \theta_{n,m} \leq 2\pi \). Since \( u_U^n u_U^m = e^{i\theta_{n,m}} I \), the condition for the operators acting on Alice’s Hilbert space is given by

\[ P_U^n P_U^m = P_U^n v_U^n v_U^m^\dagger P_U^m e^{i\theta_{n,m}}. \]  

(21)

This equation indicates that there exists a unitary operator \( v_U \) such that \( P_U^n v_U^n = P_U^n v_U^n \) and \( P_U^m v_U^m = e^{i\theta_{n,m}} P_U^m v_U^m \). By using this unitary operator \( v_U \) that is independent of \( n \), we can rewrite (16) as

\[ (P_U^n \otimes I) U = (P_U^n v_U^n) \otimes u_U^n \]  

(22)

and

\[ (P_U^m \otimes I) U = (P_U^m v_U^m) \otimes u_U^m. \]  

(23)

The ranges of the projectors \( P_U^n \) and \( P_U^m \) form subspaces. We denote the subspace specified by the projector \( P_U^n \) as \( \mathcal{W}_U^n \) and the subspace specified by the projector \( P_U^m \) as \( \mathcal{W}_U^m \). We construct
the sum-space $\mathcal{W}_{U}^{n,m}$ of these two subspaces $\mathcal{W}_{U}^{n}$ and $\mathcal{W}_{U}^{m}$ and denote the projector whose range is this sum-space as $P_{U}^{n,m}$. Since the sum-space contains two subspaces $\mathcal{W}_{U}^{n}$ and $\mathcal{W}_{U}^{m}$, $P_{U}^{n} P_{U}^{m} = P_{U}^{n}$ and $P_{U}^{m} P_{U}^{m} = P_{U}^{m}$ are satisfied. From (22) and (23), for any vector $|\phi\rangle \in \mathcal{W}_{U}^{n,m}$, we have

$$((\phi) \otimes I)U = (\phi) v_{U} \otimes u_{U}^{\dagger}, \quad (24)$$

which proves that

$$(P_{U}^{n,m} \otimes I)U = P_{U}^{n,m} v_{U} \otimes u_{U}^{\dagger}. \quad (25)$$

Thus, we can ‘combine’ the two non-orthogonal projectors $P_{U}^{n}$ and $P_{U}^{m}$ into a single projector $P_{U}^{n,m}$.

We repeat this procedure on the set of projectors $\{P_{U}^{n}\}$ until all elements are orthogonal to each other. By renaming the resulting mutually orthogonal projectors with $P_{U}^{k}$, we obtain a set of orthogonal projectors $\{P_{U}^{k}\}$. The number of elements of $\{P_{U}^{k}\}$ is strictly smaller than that of $\{P_{U}^{n}\}$ due to the combining procedure. Thus, we have shown that if $U$ is LOCC one-piece relocalizable by a non-projective measurement, then there always exists a LOCC one-piece relocalization protocol using a projective measurement. Therefore, the necessary condition for a global unitary operation $U$ to be LOCC one-piece relocalizable using general measurements is the same as the case using the projective measurement given by (19), namely, local unitary equivalent to a controlled-unitary operation.

On the other hand, the sufficient condition for $U$ to be LOCC one-piece relocalizable can be shown by the following construction. Consider the case where a global unitary operation $U$ is local unitary equivalent to a controlled-unitary operation, i.e. $U = u_{A} \otimes u_{B} (\sum_{m} P_{A}^{m} \otimes u^{m})v_{A} \otimes v_{B}$, where $u_{A}$, $u_{B}$, $v_{A}$, $v_{B}$ and $u^{m}$ are local unitary operations and $P_{A}^{m}$ is a projector. The delocalization of the two pieces of quantum information $|\psi_{A}\rangle \otimes |\psi_{B}\rangle$ is given by performing the global unitary operation $U$. Then Alice performs a general measurement described by $\{P_{A}^{m}u_{A}^{\dagger}\}$ on her qudit and communicates the measurement outcome $m$ to Bob. Then Bob performs $(u_{B}u^{m}v_{B})^{\dagger}$ depending on Alice’s measurement outcome $m$. Bob’s final state becomes $|\psi_{B}\rangle$; therefore, LOCC one-piece relocalization has been achieved. Thus, we have proved that a global unitary operation $U$ is LOCC one-piece relocalizable if and only if $U$ is local unitary equivalent to a controlled-unitary operation.

### 5. Conclusion and discussions

In this paper, we investigated how two pieces of localized quantum information represented by the tensor product of two unknown qudit states are delocalized out of the original Hilbert spaces by performing two-qudit global unitary operations. To characterize the delocalization power of global unitary operations on quantum information, we analyzed the necessary and sufficient condition to deterministically relocalize one of the two pieces of quantum information to its original Hilbert space by using only LOCC. We proved that LOCC one-piece relocalization is possible if and only if the global unitary operation is local unitary equivalent to a controlled-unitary operation given by (20), by reducing general LOCC protocols for one-piece relocalization into one-way LOCC protocols and also a general measurement used in the one-way LOCC protocols into a projective measurement.

For any global unitary operation that is not local unitary equivalent to controlled-unitary operations, our result implies that an additional non-local resource (entanglement) on top...
of LOCC is required to perform the one-piece relocalization. We can interpret this result such that global unitary operations have more delocalization power on quantum information than controlled-unitary operations; therefore, they strongly ‘intertwine’ the two pieces of quantum information such that LOCC cannot untwine even sacrificing a piece of quantum information.

Our result also reveals that the delocalization power and the entangling power [10] characterize different non-local properties of global unitary operations. To see this, we consider a two-qubit global unitary operation given by

\[
\exp\left( i\alpha \sum_{j=x,y,z} \sigma_A^j \otimes \sigma_B^j \right),
\]

which is in the form of decomposition proposed by [10], where \(\sigma_A^j\) and \(\sigma_B^j\) \((j = x, y, z)\) are Pauli operators on Alice’s Hilbert space and Bob’s Hilbert space, respectively. The operator Schmidt rank [14] of this global unitary operation is known to be four unless \(\alpha = 0\), and that of a controlled-unitary operation is known to be two unless it is the identity operation. Two-qubit unitary operations are local unitary equivalent to each other if and only if their operator Schmidt ranks coincide. Thus, the global unitary operation given by (26) for an arbitrary non-zero \(\alpha\) is not LOCC one-piece relocalizable, and belongs to the higher class in terms of the delocalization power. The entangling power depends on the parameter \(\alpha\) and, as \(\alpha\) tends to zero, it reaches zero. So for very small \(\alpha\), the entangling power of (26) can be arbitrarily close to zero. On the other hand, a CNOT (controlled-\(\sigma_Z\)) operation given by \(|0\rangle\langle0| \otimes \mathbb{I} + |1\rangle\langle1| \otimes \sigma_B^Z\) is LOCC one-piece relocalizable; therefore, it belongs to the lower class in terms of the delocalization power, but it has higher entangling power (1-ebit) than (26) for very small \(\alpha\).

We note that for any two-qubit controlled-unitary operation, the control qubit can be switched between Alice and Bob by local unitary operations; hence LOCC one-piece relocalizability is possible at either Alice’s or Bob’s side. On the other hand, for general qudits \((d \geq 3)\), controlled-unitary operations do not necessarily exhibit this bi-directional LOCC one-piece relocalizability. To see this, let us consider a two-qutrit controlled-unitary operation \(|0\rangle\langle0| \otimes \mathbb{I} + |1\rangle\langle1| \otimes Z_3 + |2\rangle\langle2| \otimes X_3\). Here, \(Z_3\) and \(X_3\) are generalized Pauli \(X\) and \(Z\) operators, respectively, given by \(Z_3 = \sum_{j=0,1,2} \exp(2\pi j/3) |j\rangle\langle j|\) and \(X_3 = |1\rangle\langle0| + |2\rangle\langle1| + |0\rangle\langle2|\). The quantum information cannot be LOCC relocalizable at Alice’s side since \(Z_3\) and \(X_3\) are not simultaneously diagonalizable.

It is also possible to further quantify the delocalization power of global unitary operations that are not local unitary equivalent to controlled-unitary operations, by evaluating how much additional ‘non-localness’ is required to perform the one-piece relocalization map. One way to quantify such non-localness is to use the ‘entangling power’ \(f_{ep}(\Gamma)\) of a map \(\Gamma\), i.e.

\[
f_{ep}(\Gamma) = \max_{\rho_{AB} \in S(H_A \otimes H_B)} E(\Gamma(\rho_{AB})) - E(\rho_{AB}),
\]

where \(E(*)\) is some entanglement measure. Another possible way is to consider the implementation of one-piece relocalization via entanglement-assisted LOCC and evaluate the minimum amount of assisting entanglement. More formally, this ‘entanglement cost’ \(f_c(\Gamma)\) for entanglement-assisted LOCC implementation of a map \(\Gamma\) is defined by

\[
f_c(\Gamma) = \min\{E(\rho_{res})^2 \Lambda_{LOCC} \text{ s.t. } Tr_{res} \Lambda_{LOCC}(\rho_{AB} \otimes \rho_{res}) = \Gamma(\rho_{AB})\},
\]

\[
\]
where $\rho_{\text{res}}$ denotes the assisting entanglement resource shared between the two parties for implementing the map $\Gamma$. These quantifications may also be used to characterize approximate relocalization, which was not considered in this work. We leave this direction of analysis for future works.

Finally, we stress that our result is on the delocalization power of global unitary operations for two pieces of input quantum information. As we have described in section 1, if Alice’s qudit state is a known pure state and we investigate the delocalization power of a global unitary operation only on Bob’s piece of input quantum information, this problem corresponds to the work of Gregoratti and Werner [11]. They have shown that the reduced map (channel) for the one piece of quantum information has to be a random unitary channel. The Stinespring extension of this random unitary channel can be described by using a controlled-unitary operation. However, it is not necessary for the global unitary operation used in the Stinespring extension of this channel to be a controlled-unitary operation. For example, it has been shown that a two-qubit unitary operation given by $U = |0⟩⟨0| \otimes |0⟩⟨0| + |0⟩⟨1| \otimes |1⟩⟨0| + |1⟩⟨0| \otimes |0⟩⟨1| − |1⟩⟨1| \otimes |1⟩⟨1|$ is not local unitary equivalent to a controlled-unitary operation in [15]. But if we fix Alice’s state as $|+⟩ = (|0⟩ + |1⟩)/\sqrt{2}$, LOCC one-piece relocalization can be achieved by Alice’s projective measurement $\{|+⟩⟨+|, |−⟩⟨−|\}$ followed by Bob’s local unitary operation $H$ when Alice’s outcome is $+$, or $σ_0^z \cdot H$ when it is $−$, where $H$ denotes the Hadamard gate given by $H = |0⟩⟨+| + |+⟩⟨0|$. Therefore, the delocalization power of global unitary operations depends on the number of pieces of input quantum information.

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